MTH 310 HW 10 Solutions

April 18, 2016

Section 5.2, Problem 8

Write out the multiplication table for $\mathbb{Q}[x]/(x^2)$.
Answer. If $[p(x)], [q(x)] \in \mathbb{Q}[x]/(x^2)$, then $[p(x)] = [ax + b]$ and $[q(x)] = [cx + d]$. Then $[p(x)][q(x)] = [p(x)q(x)] = [acx^2 + (ad + bc)x + bd] = [(ad + bc)x + bd]$.

Section 5.2, Problem 11

Show that the above ring is not a field.
Answer. In $\mathbb{Q}[x]/(x^2)$, $[x][x] = [x^2] = [0]$, so there is a nonzero divisor of zero. The claim then follows by this theorem, which is proven differently in Hungerford.
Theorem 1. All fields are integral domains. That is, if a, b are elements of a field and $ab = 0$, then either $a = 0$ or $b = 0$.

Proof. Assume the above with $a \neq 0$. Multiplying both sides of the equation $ab = 0$ by a^{-1} (which exists as $a \neq 0$) yields $b = 0$. □

Note that there are rings in which $ab = 0$ but both a, b are not zero. For example, in $\mathbb{Z}\backslash(6)$ (integers modulo 6), $2 * 3 = 0$. Pretty crazy.

1 Section 5.3, Problem 3

If $a \in \mathbb{F}$, describe the field $\mathbb{F}[x]/(x - a)$.
Answer. We claim that $\mathbb{F}[x]/(x - a)$ is isomorphic to $\mathbb{F}$. To show this, let $f : \mathbb{F} \to \mathbb{F}[x]/(x - a)$ where $f(z) = [z]$. We claim f is an isomorphism. It is a homomorphism, and is surjective since every class $[p(x)] \in \mathbb{F}[x]/(x - a)$ has an element $q \in \mathbb{F}$, by the division algorithm. It is injective because if $[z_1] = [z_2]$ then $z_1 = z_2$.